TECHNICAL
RESEARCH
REPORT

SYSTEMS
RESEARCH
CENTER

Supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

Eulerian Many-Body Problems

by P.S. Krishnaprasad

TR 89-15

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **1989**	2. REPORT TYPE	3. DATES COVERED **00-00-1989 to 00-00-1989**
4. TITLE AND SUBTITLE **Eulerian Many-Body Problems**		5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of Maryland,Systems Research Center,College Park,MD,20742**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT **see report**		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**		**23**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

89-15

Contemporary Mathematics
Volume 97, 1989

EULERIAN MANY-BODY PROBLEMS

P.S. Krishnaprasad *

ABSTRACT. The hamiltonian dynamics of coupled structures is discussed. There are geometric parallels in earlier work on the Newtonian (gravitational) many- body problem. In the study of relative equilibria, a theorem due to Smale has a useful role. Relative stability modulo a group of symmetries can be determined using the energy-Casimir (or energy - momentum) method. For nongeneric values of momenta, the Poisson structure can affect stability.

1. INTRODUCTION. The central role of the Newtonian (gravitational) many-body problem in celestial mechanics has inspired major advances in mathematics and physics. For an exposition see (Abraham and Marsden [1]) and (Smale [31]). In recent years, engineering applications have brought to the forefront, questions concerning the dynamics of systems of kinematically coupled structures composed of rigid and flexible bodies. We refer to these as Eulerian many-body problems to emphasize the role of Euler forces (or frame forces) in determining the interactions. Eulerian many-body problems arise as models of robotic manipulators, high speed mechanical machinery, complex spacecraft with articulated components, space-based sensors etc. See (Wittenburg [36]) and [8], [12] for expositions of engineering aspects and basic formulations of underlying models.

In recent work, [13] [17] [18] [26] [27] [29] [33], we have explored the rich geometry of Eulerian many-body problems. We have used the geometry of symplectic manifolds, Poisson structures, and reduction by symmetry groups in creating a framework for the study of the dynamic behavior of certain classes of Eulerian many-body problems. Among the classes of problems we have investigated, we include rigid bodies carrying rotors, planar many- body systems, three dimensional systems coupled by ball and socket joints, and rigid bodies with flexible attachments modeled by geometrically exact formulations of elasticity. Our methods shed light on questions regarding relative equilibria, periodic orbits, stability,

1980 Mathematics Subject Classification (1985 Revision): 58F05, 58F10
* This work was supported in part by the AFOSR University Research Initiative Program under grant AFOSR- 87-0073 and by the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012.

conservation laws (e.g. Casimir functions) and controllability on level sets of conservation laws.

The present paper simply highlights some key geometric aspects of these later developments.

ACKNOWLEDGEMENT

This is based on a stimulating collaboration with Robert Grossman, Jerrold Marsden, Yong-Geun Oh, Tom Posbergh, Juan Simo and N. Sreenath, and their contributions are gratefully acknowledged. We have also benefited from conversations with John Baillieul, Anthony Bloch, Mark Levi, Debra Lewis and Tudor Ratiu. A special thanks to Jerrold Marsden for his enthusiastic support over the years as friend, collaborator and teacher.

2. GEOMETRY

The abstract framework for Eulerian many-body problems is the one isolated by Smale in his study of the gravitational many-body problem. Let (M, K) be a Riemannian manifold and let G be a Lie group with associated action,

$$\begin{aligned} \Phi : G \times M &\to M \\ (g, q) &\mapsto \Phi_g(q) \end{aligned}$$

where Φ_g is an isometry for all $g \in G$. The Riemannian metric induces a vector bundle isomorphism

$$K^b : TM \to T^*M$$

defined by

$$K^b(v_q) \cdot w_q = K(v_q, w_q), \text{ forall } v_q, w_q \in TM_q \cdot$$

The canonical symplectic structure $\omega = -d\theta_0$ on T^*M can be pulled back to

$$\Omega = (K^b)^*(\omega),$$

also an exact symplectic structure on TM. The action Φ lifts to symplectic actions $T\Phi$ and $T\Phi^*$ on TM and T^*M respectively.

Let $V : M \to \mathbb{R}$ be a G-invariant (potential) function on M. The hamiltonian $H : T^*M \to \mathbb{R}$, is defined by,

$$H\,(\alpha_q) \;=\; \frac{1}{2}\,K\;\;\Big((K^{\flat-1})\;\alpha_q,\;(K^{\flat-1})\alpha_q\Big)$$
$$+\;V_\circ\tau_M^*\,(\alpha_q)$$

where $\tau_M \;:\; T^*M \;\to\; M$ is the canonical projection.

Associate to H a vector field X_H on T^*M by requiring that,

$$dH\,(Y) \;=\; \omega\,(X_H, Y)$$

for all vector fields Y on T^*M. The hamiltonian system $(T^*M,\;\omega,\;X_H)$ is a simple mechanical system with symmetry in the sense of Smale. It admits a momentum mapping in a natural way. To see this, let $\Im$ denote the Lie algebra of G and $\Im^*$ the dual space of $\Im$. The symplectic action $T\Phi^*$ on T^*M, defines a Lie algebra homomorphism, of $\Im$ into hamiltonian vector fields on T^*M; we denote this correspondence as $\xi \;\mapsto\; \xi_{T^*M}$. Then the map,

$$J\;:\;T^*M\;\to\;\Im^*$$

defined by,

$$J\,(\alpha_q)\cdot\xi \;=\; (i_{\xi_{T^*M}}\,\theta_0)\,(\alpha_q)\;,\quad \xi\in\Im$$

is an Ad^* - equivariant momentum mapping. Hence J is a conserved quantity of the system (T^*M, ω, X_H) .

The framework sketched so far is the proper setting for Eulerian many-body problems in our sense (as it is for Smale's approach to the gravitational many-body problem).

EXAMPLE 1 (Planar two-body problem)

Imagine two rigid laminae connected by a pin joint, floating in a gravity-free planar universe (see figure 1). For an observer at the center of mass of the system of two bodies, the absolute orientations of the two bodies, determined say by attaching body-frames, are sufficient to determine the absolute configuration of the pair. The group S^1 of spatial rotations of the observer's frame is a symmetry group for the problem. Thus, $M \;=\; S^1 \;\times\; S^1$, $G \;=\; S^1$ acting on M via the diagonal action and the metric on M is given by

$$\begin{aligned} K\,(\dot\theta_1,\;\dot\theta_2) &= 2\;\times\;\text{Kinetic energy}\\ &= \left\langle \begin{bmatrix} \tilde I_1 & \lambda\,(\theta)\\ \lambda\,(\theta) & \tilde I_2 \end{bmatrix} \begin{pmatrix}\dot\theta_1\\ \dot\theta_2\end{pmatrix},\;\begin{pmatrix}\dot\theta_1\\ \dot\theta_2\end{pmatrix}\right\rangle\\ &= \langle \mathbf{I}_p\omega,\;\omega\rangle \end{aligned}$$

where, $\tilde I_i \;=\; I_i \;+\; \epsilon\;d_i^2,\; i \;=\; 1,2,$ are augmented inertias of the bodies, $\epsilon \;=\; m_1\,m_2\;/\;(m_1+m_2)$ is a reduced mass, and for the choice of body frames as in figure 1,

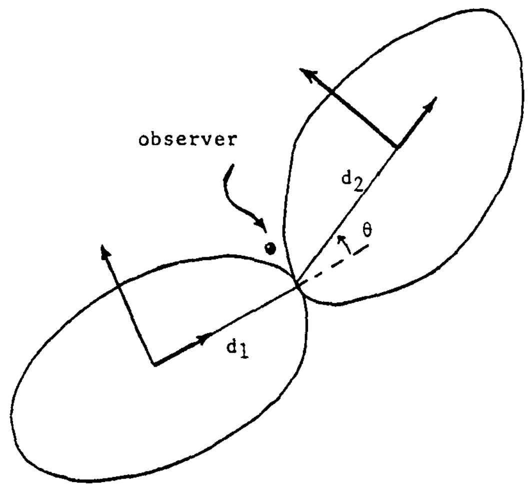

Figure 1. Planar Two-Body Problem

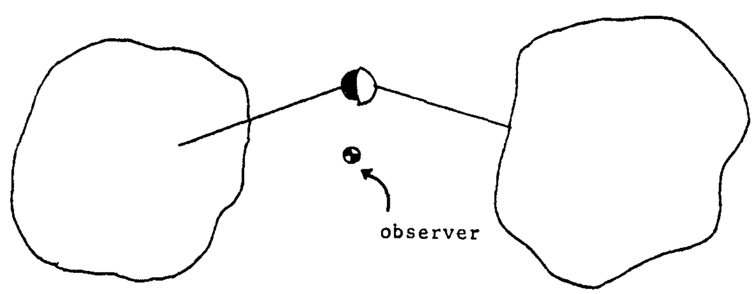

Figure 2. Rigid Bodies Coupled by a Ball and Socket Joint

$\lambda(\theta) = \epsilon\, d_1 d_2 \sin(\theta_1 - \theta_2)$ is a function of the joint angle. Since K depends only on the difference $\theta_1 - \theta_2$, it is invariant under the S^1 action $(\theta_1, \theta_2) \mapsto (\theta_1 + g, \theta_2 + g)$, $g \epsilon S^1$. The subscript in $\mathbf{I}_p$ is in reference to planarity. The vector bundle map K^b is given by

$$K^b(\omega) = \mu = \mathbf{I}_p \omega.$$

The momentum mapping for the S^1 action is then,

$$J_p : T^*(S^1 \times S^1) \to \mathbb{R}$$

$$(\theta_1, \theta_2, \mu_1, \mu_2) \mapsto \mu_1 + \mu_2$$

It is just the angular momentum of the system with respect to the observer at the center of mass.

EXAMPLE 2. (Rigid bodies coupled by a ball and socket joint)

This is a spatial analog of the previous planar example. The two bodies are free to move in three dimensions, subject to a (three degrees of freedom) ball and socket coupling. As before, the observer is at the center of mass of the system of two bodies. See figure 2 below for a representation.

In this case $M = SO(3) \times SO(3)$ and $G = SO(3)$ acts diagonally on M

$$\begin{aligned} \Phi : SO(3) \times M &\to M \\ (P, A_1, A_2) &\mapsto (PA_1, PA_2). \end{aligned}$$

This is just the symmetry associated to the freedom of the observer to make arbitrary spatial rotations of his frame.

The action Φ leaves the kinetic energy metric invariant, the latter given by a 6×6 positive definite quadratic form $\mathbf{I}_s$ analogous to $\mathbf{I}_p$ in example 1, with only off-diagonal terms dependent on configurations. For $SO(3)$ invariance, these in fact depend only on $A_1^{-1} A_2$ the relative configuration of two bodies. Once again an Ad^*- equivariant moment mapping $J_s : T^*(SO(3) \times SO(3)) \to SO(3)^* \simeq \mathbb{R}^3$ can be written down. It is just the angular momentum of the system with respect to the observer at the center of mass.

In the thesis of Sreenath and in the papers by Oh, Sreenath, Marsden and Krishnaprasad, planar coupled systems such as that in example 1 are investigated.

In the paper of Grossman, Krishnaprasad and Marsden the example 2 is discussed.

For the most part, in these references the situations analyzed require that the potential $V \equiv 0$. However, in Sreenath's thesis, control functions at the joints of planar many-body

systems are considered and the associated feedback laws may in certain cases be interpreted as arising from potential functions due to torsional springs at the joints.

Other interesting examples including flexible bodies (attachments) appear in [18], [27], [30], and in the papers of Baillieul and Levi [5] [6] [7].

Poisson structures are central to our point-of-view. A Poisson manifold P is simply a smooth manifold equipped with an $\mathbb{R}$ -bilinear map (Poisson structure),

$$\{\cdot,\cdot\}_P : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$$

satisfying the axioms

(i) $\{f,g\}_P = -\{g,f\}_P$

(ii) $\{fg,h\}_P = g\{f,h\}_P + f\{g,h\}_P$

(iii) $\{f, \{g,h\}_P\}_P + \{g,\{h,f\}_P\}_P + \{h,\{f,g\}_P\}_P = 0.$

We outline the general theory a little before we specialize to the mechanical setting. First, associated to a Poisson structure, there is a unique, twice contravariant skew-symmetric, smooth tensor field Λ on P such that,

$$\{f,g\}_P = \Lambda\,(df,dg).$$

For a proof see p. 109 of [19]. The tensor Λ defines a vector-bundle morphism,

$$\begin{aligned} \Lambda^\# : T^*P &\to TP \\ \alpha_x &\mapsto \Lambda^\#(\alpha_x) \in TP_x \end{aligned}$$

satisfying,

$$\beta_x\,(\Lambda^\#(\alpha_x)) = \Lambda\,(x)\,(\alpha_x,\beta_x) \text{ for all } \beta_x \in TP_x.$$

The rank of the Poisson structure at $x \in P$ is defined to be the rank of the Poisson tensor Λ at x. This is simply the rank of the (characteristic) distribution $C = \Lambda^\#(T^*M) \subset TM$ at the point x. The rank may vary on P. However, it is a theorem of Kirillov [16] that $\Lambda^\#(T^*M)$ defines a generalized foliation on P such that through each point $x \in P$, passes a leaf carrying a unique symplectic structure that makes the injection map of that leaf a Poisson morphism. (See Weinstein [34] and Libermann-Marle [19]). Thus a Poisson manifold is a union of symplectic leaves.

A function $f \in C^\infty(P)$ is called a Casimir function if

$$\{f,g\}_P = 0 \quad \forall\, g \in C^\infty(P).$$

Casimir functions are constant on symplectic leaves.

Let G be a Lie group and let $\Psi : G \times P \to P, (g,x) \mapsto \Psi_g(x)$, be a group action such that, $\Psi_g(\cdot)$ is a Poisson morphism for every $g \in G$. Further, suppose that the action is proper and free. Then there exists a good quotient P/G that carries a Poisson structure $\{\cdot,\cdot\}_{P/G}$ induced from the one on P satisfying,

$$\{f,g\}_{P/G} = \{f \circ \pi, g \circ \pi\}_P.$$

Here $\pi : P \to P/G$ is the canonical projection. By construction, it is a Poisson morphism.

G-equivariant dynamics on P induce dynamics on P/G. Suppose $h : P \to \mathbb{R}$ is a G-invariant hamiltonian function on P , i.e.,

$$h(\Psi_g(x)) = h(x) \ \forall\, g \in G.$$

Define a vector field X_h by

$$X_h f = \{f,h\}_P \ \forall\, f \in C^\infty(P).$$

The hamiltonian h descends to $\hat{h} : P/G \to \mathbb{R}$ and determines a reduced dynamics $\hat{X}_{\hat{h}}$ on P/G by

$$\hat{X}_{\hat{h}}(\hat{f}) = \{\hat{f},\hat{h}\}_{P/G} \ \forall \hat{f} \in C^\infty(P/G).$$

Here $\hat{h}([x]) = h(x)$ for any equivalence class $[x]$ in P/G. From, the definition of the characteristic distribution $C = \Lambda^{\#}(T^*M)$, it follows that the hamiltonian vector fields $\hat{X}_{\hat{h}}$ leave invariant the symplectic leaves. Thus any Casimir function is an integral of motion for $\hat{X}_{\hat{h}}$. The trajectories of X_h project under π to trajectories of $\hat{X}_{\hat{h}}$. The steps just outlined constitute the essence of Poisson reduction. See [22] for more details. We give some examples of Poisson structures.

EXAMPLE 3. (P,ω) is a connected symplectic manifold and $\{f,g\}_P := \omega(X_f,X_g)$.

Here the rank = dimension of P and there is just one symplectic leaf. Simple mechanical systems with symmetry yield interesting rank-degenerate cases. Referring to example 1 (the planar two-body problem), set $(P,\omega) = \left(T^*(S^1 \times S^1),\omega\right)$. The diagonal S^1 action, being symplectic, also leaves the Poisson structure on $T^*(S^1 \times S^1)$ invariant. The Poisson-reduced phase space $\left(T^*(S^1 \times S^1)\right)/S^1$ has a bracket structure

$$\{f,g\} = \left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \mu_1} - \frac{\partial f}{\partial \mu_1}\frac{\partial g}{\partial \theta_2}\right) - \left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \mu_2} - \frac{\partial f}{\partial \mu_2}\frac{\partial g}{\partial \theta}\right)$$

where $\theta = \theta_1 - \theta_2 =$ joint angle.

Symplectic leaves on $\left(T^*(S^1 \times S^1)\right)/S^1$ are cylinders (the corresponding characteristic distribution is of rank 2 everywhere) and are level sets of the Casimir function $\phi(\mu_1, \mu_2, \theta) = \mu_1 + \mu_2$.

EXAMPLE 4 (dual space of $\Im$)

$\Im^*$ carries the Lie-Berezin-Kirillov-Kostant-Souriau Poisson structure (s), defined by

$$\{f,g\}_{\mp}(\mu) = \mp\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle$$

where $f, g \in C^\infty(\Im^*)$ and $\mu \in \Im^*$. The minus (plus) bracket is obtained by viewing $\Im^*$ as the left (right) Poisson reduction of T^*G by G.

The symplectic leaf through μ is $\mathcal{O}_\mu = \left\{\ell \in \Im^* : \ell = Ad^*_{g^{-1}}(\mu), g \in G\right\}$ the coadjoint orbit through μ.

When $\Im = so(3)$, the Poisson structure on $\Im^*$ is of rank 2 everywhere except at the origin where it is of rank 0.

We close this section with some remarks about dual pairs. Given a symplectic manifold S and Poisson manifolds P_1, P_2, suppose maps J_1 and J_2 can be found such that the following is a diagram of Poisson morphisms:

$$P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$$

The diagram is a <u>dual pair</u> in the sense of Marsden and Weinstein [24] [34] if the function algebras $\mathcal{F}_1 = J_1^*(C^\infty(P_1))$ and $\mathcal{F}_2 = J_2^*(C^\infty(P_2))$ are polar i.e.,

$$\{\mathcal{F}_1, \mathcal{F}_2\} = 0.$$

In that case the Casimir functions on P_1 and P_2 are in one-to-one correspondence and to the space $\mathcal{F}_1 \cap \mathcal{F}_2$.

Suppose the G action Φ on a simple mechanical system with symmetry is proper and free. Then there is an associated dual pair,

$$\Im^* \xleftarrow{J} T^*M \xrightarrow{\pi} T^*M/G.$$

Let $\mathcal{O}_\mu$ be the coadjoint orbit through $\mu \in \Im^*$ and let G_μ = isotropy subgroup of μ under the coadjoint action. Then $\mathcal{O}_\mu \simeq G/G_\mu$. Furthermore, the symplectic leaves in T^*M/G are the manifolds $\pi\ (J^{-1}(\mathcal{O}_\mu)) = J^{-1}(\mathcal{O}_\mu)/G$. They are isomorphic to the Marsden - Weinstein - Meyer spaces of symplectic reduction [23].

3. RELATIVE EQUILIBRIA. Much work on the gravitational many-body problem has concentrated on special uniformly rotating configurations (e.g. Moulton's theorem on collinear configurations). These are relative equilibria. The search for relative equilibria in Eulerian many-body problems has yielded some interesting results [33] [26].

Consider the dual pair

$$\Im^* \xleftarrow{J} (S,\omega) \xrightarrow{\pi} S/G$$

and $h : S \to \mathbb{R}$ a hamiltonian invariant under the action of G

DEFINITION. $z_e \in S$ is a <u>relative equilibrium</u> (or the flow $F^t_{X_h}\ (z_e)$ is a <u>stationary motion</u>) if there exists $\xi \in \Im$ such that

$$F^t_{X_h}\ (z_e) = \Psi\ (exp\ (t\xi),\ X_e).$$

THEOREM (Relative Equilibrium)

The following are equivalent:

(i) z_e is a relative equilibrium;

(ii) z_e is a critical point of $h_\xi = h - < J, \xi >$, for some $\xi \in \Im$;

(iii) $\pi\ (z_e)$ is an equilibrium for the dynamics $\hat{X}_{\hat{h}}$ on S/G ∎

REMARK. See Abraham & Marsden, chapter 4, for proofs. Part (ii) above is also a consequence of the Souriau-Smale-Robbin theorem.

For simple mechanical systems with symmetry, there is an elegant characterization of relative equilibria (due to Smale [31], although special versions have been known earlier).

THEOREM (Smale). Consider a simple mechanical system with symmetry (T^*M, ω, X_H) as defined in Section 2. Define,

$$V_\xi : M \to \mathbb{R}$$
$$q \mapsto V(q) - \frac{1}{2} K\ (\xi_M(q), \xi_M(q))$$

for each $\xi \in \Im$.

Then $z_e = (q_e, p_e) \in T^*M$ is a relative equilibrium iff, q_e is a critical point of V_ξ for some $\xi \in \Im$ and $p_e = K^b(\xi_M(q_e))$. ∎

For a proof of Smale's theorem see Smale [31] or Abraham & Marsden, pp 355. In his well-known paper [31], Smale uses this theorem to prove Moulton's theorem on the number of collinear configurations for the gravitational many-body problem.

Smale's theorem provides a convenient technique to compute relative equilibria. V_ξ is a G_ξ-invariant function on M the configuration space, where

$$G_\xi = \{g \in G : Ad_g(\xi) = \xi\}$$

and we are in the setting of equivariant Morse theory [3].

EXAMPLE 5. (planar 2-body problem continued). Returning to examples 1 and 3, we note, for $\xi \in \Im \equiv \mathbb{R}$, ξ_M is given by

$$\xi_M((\theta_1, \theta_2)) = \xi\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right).$$

Thus, setting $V \equiv 0$,

$$\begin{aligned} V_\xi((\theta_1, \theta_2)) &= -\frac{\xi^2}{2}(1,\ 1)\mathbf{I}_p\left(\begin{pmatrix}1\\1\end{pmatrix}\right) \\ &= -\frac{\xi^2}{2}\left(\tilde{I}_1 + \tilde{I}_2 + 2\lambda(\theta_1 - \theta_2)\right). \end{aligned}$$

The S^1-equivalence classes of critical points of V_ξ are given by,

$$\frac{d\lambda}{d\theta} = 0 \leftrightarrow \theta = 0 \text{ or } \pi.$$

More generally, for a chain of n planar laminae, one expects at least $(1+1)^{n-1} = 2^{n-1}$ relative equilibrium classes since the Poincare' polynomial of the $(n-1)$ torus is $(1+t)^{n-1}$.

We add that L-S. Wang, at the University of Maryland, has begun a numerical search for stable relative equilibria in the ball and socket problem of example 2 by numerical minimization of V_ξ.

4. RELATIVE STABILITY MODULO G. In the presence symmetries, a natural notion of stability is the following.

DEFINITION. Let $z_e \in S$ be a relative equilibrium for the dynamics X_h corresponding to a G-invariant hamiltonian h on (S, ω). We say that z_e is relatively stable modulo

G if $\pi(z_e)$ is a Lyapunov stable equilibrium for the Poisson reduced dynamics $\hat{X}_{\hat{h}}$ on S/G.

There is a sufficient condition for relative stability modulo G.

THEOREM (Relative Stability). $\pi(z_e)$ is an equilibrium point of $\hat{X}_{\hat{h}}$ *iff* it is a critical point of $\hat{h}_{|L}$ the restriction of $\hat{h}$ to the symplectic leaf L through $\pi(z_e)$. In that case, $\pi(z_e)$ is Lyapunov stable if,

(i) the Hessian $D^2(\hat{h}_{|L})(\pi(z_e))$ is definite.

(ii) the point $\pi(z_e)$ has a neighborhood W on which the rank of the Poisson structure $\{\cdot\,,\,\cdot\}_{S/G}$ is constant.

REMARKS. In the form stated, the relative stability theorem appears to be due to Arnold. See also [19], Theorem 12.4 in chapter III. Points in S/G satisfying condition (ii) are called generic points. At generic points, nontrivial (local) Casimir functions C_ϕ exist. One can verify condition (i) by seeking a (local) Casimir C_ϕ such that $\pi(z_e)$ is an unconstrained critical point of $\hat{h} + C_\phi$ and $D^2(\hat{h} + C_\phi)$ at $\pi(z_e)$ is definite. This is the essence of the <u>energy - Casimir</u> method. Equivalently one can find $\xi \in \Im$ such that $dh_\xi(z_e) = 0$ and $D^2 h_\xi(z_e)$ is definite in directions transversal to neutral directions associated to G_ξ. This is the essence of the <u>energy - momentum</u> method. For simple mechanical systems with symmetry $(S = T^*M)$, using Smale's theorem of section 3 and a splitting of $T(T^*M)$, this reduces to checking $D^2V_\xi(q_e)$ is positive definite (see the paper of Marsden and Simo in this volume).

At nongeneric points (where condition (ii) above does not hold), one may, by <u>ad hoc</u> methods find conserved Lyapunov functions for $\hat{X}_{\hat{h}}$. But there exist examples due to Weinstein [35] and Libermann - Marle [19] indicating that at nongeneric points in S/G, definiteness of $D^2\,\hat{h}_{|L}$ does not imply stability. The Poisson bracket in $\{\cdot\,,\,\cdot\}_{S/G}$ can affect relative stability modulo G. See the appendix to this paper for details of an example due Libermann and Marle.

We must add that we are aware of no "physically motivated" example that parallels the one in the appendix. It would be interesting to explore this further.

EXAMPLE 6. (Planar 2-body Problem)

By energy-Casimir the stretched out relative equilibrium $(\theta = 0)$ is relatively stable mod S^1 and the folded over relative equilibrium $(\theta = \pi)$ is unstable. This is true even at zero total angular momentum since the Poisson tensor is of constant rank 2.

5. HOLONOMY. In 1987, Jair Koiller introduced us to the concept of Berry's geometric phase. Inspired by his remarks, we worked out a formula for planar n-body chains that

admits interpretation via holonomy of a connection.

Consider a chain of planar rigid bodies floating in a planar gravity-free universe as in figure 3. Suppose each joint is actuated so as to permit free adjustment of joint angles. Assume that the whole assembly is at rest (angular momentum =0).

PROBLEM. Suppose the joint angles are varied continuously in a prescribed manner and brought back to their initial condition of rest. What will be the displacement of body 1 from its initial absolute orientation?

In geometric terms, a loop is traversed in T^{n-1} the joint angle space (or labelled shape space in the terminology of R. Montgomery) and we are interested in measuring the holonomy or extent to which it fails to lift to a loop in the absolute configuration space T^n. Such liftings require connections [20] and there is a natural one in the problem obtained by taking the orthogonal complement of the subspace spanned by vertical vector fields. Postponing the details to a future publication we would like to give a formula answering the problem above.

Let $\mathbf{I}_p^n$ denote the $n \times n$ quadratic form associated to the planar n-body system analogous to $\mathbf{I}_p$ in example 1. (see the thesis of Sreenath for explicit form of $\mathbf{I}_p^n$). Then the angular momentum relative to the observer at the center of mass is.

$$c = e \cdot \mathbf{I}_p^n \, \omega,$$

where $e = (1, 1, \cdots, 1)'$, and ω is the vector of angular velocities of the system. Admissible motions of the system leave,

$$e \cdot \mathbf{I}_p^n \, \omega = 0.$$

Then the phase shift of body 1 is given by

$$\triangle\theta_1 = -\int_\Gamma \frac{e \cdot \mathbf{I}_p^n M d\phi}{e \cdot \mathbf{I}_p^n e},$$

where $d\phi = (d\phi_1, \cdots, d\phi_{n-1})$ is the vector of joint differentials and M is an $n \times (n-1)$ matrix satisfying

$$M_{ij} = \begin{cases} 0 & i = 1 \\ 1 & i > j \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and Γ is the loop traversed in joint-angle space.

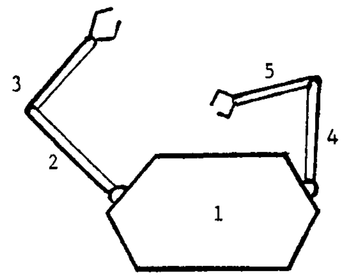

Figure 3. Planar n-Body System

The above formula can be useful in practical computations. Nonabelian analogs of this formula applicable to say the ball and socket problem can be derived from the theory of connections.

There is a related question of great interest in control theory.

PROBLEM. Among all possible parameterized paths Γ in joint angle space, find one that minimizes the action,

$$\int_\Gamma \omega \cdot \mathbf{I}_p^n \, \omega \, dt$$

and attains a prescribed phase shift $\triangle\theta_1$.

Control theoretic antecedents of this problem in the setting of Lie groups go back to the early papers of Brockett [9] and the Ph.D thesis of Baillieul [4]. The work of Brockett [10] [11] on singular Riemannian geometry and the recent results of Richard Montgomery [25] are directly applicable. We hope to report on this at a later date.

REFERENCES

1. R. Abraham and J.E. Marsden (1978), *Foundations of Mechanics*, Second Edition, revised, enlarged, reset. Benjamin/Cummings, Reading.
2. V.I. Arnold (1978), *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York.
3. M.F. Atiyah and R. Bott (1984), "The Moment Map and Equivariant Cohomology", *Topology*, vol 23, no.1, 1-28.
4. J. Baillieul (1978), "Geometric Methods for Nonlinear Optimal Control Problems", *Journal of Optimization Theory & Applications*, vol. 25, no. 4, 519-548.

5. J. Baillieul (1987), "Equilibrium Mechanics in Rotating Systems", *Proc. 26th IEEE Conf. Dec. Control,* IEEE, New York, 1429-1434.

6. J. Baillieul (1988). "Linearized Models for the Control of Rotating Beams", *Proc. 27th IEEE Conf. Dec. Control*, IEEE, New York. 1726-1731.

7. J. Baillieul and M. Levi (1987), "Rotational Elastic Dynamics", *Physica D*, 27D, 43-62.

8. G. Bianchi and W. Schielen (1985), eds. *Dynamics of Multibody Systems*, Proceedings of IUTAM/IFToMM Symposium Udine/Italy 1985, Springer Verlag, Berlin.

9. R. W. Brockett (1973), "Lie Theory and Control Systems Defined on Spheres", *SIAM Journal on Applied Mathematics*, vol. 25, no. 2, 1973.

10. R.W. Brockett (1981), "Control Theory and Singular Riemannian Geometry", in P.J. Hilton and G.S. Young eds., *New Directions in Applied Mathematics*, Springer-Verlag, Berlin, pp 11-27.

11. R.W. Brockett (1983), "Nonlinear Control Theory and Differential Geometry", in *Proc. Intl. Congr. Math.*, Warsaw, 1357-1368.

12. CIME (1972), *Stereodynamics*, edizione cremonese, Roma.

13. R. Grossman, P.S. Krishnaprasad and J.E. Marsden (1987) "The Dynamics of Two Coupled Three Dimensional Rigid Bodies" in F. Salam & M. Levi, eds. *Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits,* pp. 373-378, SIAM Publ., Philadelphia, 1988.

14. V. Guillemin and S. Sternberg (1984), *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge.

15. D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein (1984), "Stability of Rigid Body Motion using the Energy - Casimir Method", in J.E. Marsden ed. *Fluids and Plasmas: Geometry and Dynamics*, in series *Contemporary Mathematics*, vol. 28, 15-23, AMS, Providence.

16. A.A. Kirillov (1976), "Local Lie Algebras", *Russian Math. Surveys*, vol. 31, 56-75.

17. P.S. Krishnaprasad (1985), "Lie-Poisson Structures Dual-Spin Spacecraft and Asymptotic Stability", *Nonlinear Analysis : Theory, Methods and Applications*, vol. 9, no. 10, 1011-1035.

18. P.S. Krishnaprasad and J.E. Marsden (1987) "Hamiltonian Structures & Stability for Rigid Bodies with Flexible Attachments." *Arch. Rat. Mech. Anal. 98,* 71-93.

19. P. Libermann and C-M. Marle (1987), *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publ., Dordrecht.

20. J.E. Marsden, R. Montgomery and T. Ratiu (1988), "Reduction, Symmetry and Berry's Phase in Mechanics", Preprint, Cornell University.

21. J.E. Marsden, T. Ratiu and A. Weinstein (1984), "Reduction and Hamiltonian Structures on Duals of Semidirect Product Lie Algebras", in J.E. Marsden ed. *Fluids and Plasmas: Geometry and Dynamics*, in series *Contemporary Mathematics*, vol. 28, 55-100, AMS, Providence.

22. J.E. Marsden and T. Ratiu (1986), "Reduction of Poisson Manifolds", *Letters in Math. Phys.*, vol 11, 161-169.

23. J. E. Marsden and A. Weinstein (1974), "Reduction of Symplectic Manifolds with Symmetry", *Reports in Math. Phys.*, vol. 5, 121-130.

24. J.E. Marsden and A. Weinstein (1983), "Coadjoint Orbits, Vortices, and Clebsch Variables for Incompresible Fluids", *Physica 7D,* 305-323.

25. R. Montgomery (1988), "Shortest Loops with a Fixed Holonomy", Preprint, Mathematical Sciences Resarch Institute, Berkeley, MSRI 01224-89.

26. Y.G. Oh, N. Sreenath, P.S. Krishnaprasad and J.E. Marsden (1988), "The Dynamics of Coupled Planar Rigid Bodies Part II: Bifurcation, Periodic Orbits, and Chaos", (in press) *J. Dynamics & Differential Equations.*

27. T. Posbergh, P.S. Krishnaprasad and J.E. Marsden (1987), "Stability Analysis of a Rigid Body with a Flexible Attachment using the Energy-Casimir Method" in M. Luksic, C. F. Martin, W. Shadwick eds. *Differential Geometry : The Interface between Pure and Applied Mathematics*, in series, *Contemporary Math.*, vol. 68, 253-273. AMS, Providence.

28. T. Posbergh (1988) Ph.D. Thesis, University of Maryland *"Modeling and Control of Mixed and Flexible Structures"*. Also, Systems Research Center Technical Report SRC TR 88-58.

29. J.C. Simo, J.E. Marsden and P.S. Krishnaprasad (1988), "The Hamiltonian Structure of Nonlinear Elasticity: The Material and Convective Representation of Rods, Plates and Shells", *Arch. Rat. Mech. & Anal.,* vol. 104, no. 2, 125-183.

30. J.C. Simo, T. Posbergh and J.E. Marsden (1988), "Nonlinear Stability of Geometrically Exact Rods by the Energy-Momentum Method". Preprint, Stanford University, Division of Applied Mechanics.

31. S. Smale (1970) "Topology and Mechanics, I, II", *Invent. Math.*, vol. 10, 305-331 and vol. 11, 45-64.

32. N. Sreenath (1987) Ph.D. Thesis, University of Maryland *"Modeling and Control of Multibody Systems"*. Also, Systems Research Center Technical Report SRC TR 87-163.

33. N. Sreenath, Y.G. Oh, P.S. Krishnaprasad and J.E. Marsden (1988), "The Dynamics of Coupled Planar Rigid Bodies Part I: Reduction, Equilibria & Stability *Dynamics & Stability of Systems,* vol. 3, no. 1&2, 25- 49.

34. A. Weinstein (1983), "The Local Structure of Poisson Manifolds", *J. Diff. Geom.*, vol. 18, 523-557 and vol. 22; (1985), 255.

35. A. Weinstein (1984), "Stability of Poisson-Hamilton Equilibria", in J.E. Marsden ed. *Fluids and Plasmas: Geometry and Dynamics*, in series *Comtemporary Mathematics*, vol 28, 3-13, AMS, Providence.

36. J. Wittenburg (1977), *Dynamics of Multibody Systems*, B.G. Teubner, Stuttgart.

APPENDIX. On An Example of P. Libermann and C.-M. Marle

(written with the assistance of L-S. Wang)

In this appendix, we work out an example suggested by Libermann & Marle (p. 274, [1]) to investigate the notion of relative stability modulo a group G of symmetries in the sense of Liapunov. The main purpose here is to show that for nongeneric momenta, stability may depend on the Poisson structure also.

First, the symplectic manifold in this example is

$$(M,\omega) = (\mathbb{R}^4, dq^1 \wedge dp_1 + dq^2 \wedge dp_2).$$

The Lie group here is

$$\begin{aligned} G &= Aff_+(\mathbb{R}) \\ &= \left\{ \begin{array}{ll} (a,b)| & a,b \in \mathbb{R}^2 \text{ with group law} \\ & (a,b)\cdot(a',b') \\ & = (a+a', b+e^a b') \end{array} \right\}. \end{aligned}$$

G acts on (M,ω) by the following rule.

$$\begin{aligned} \Phi : G \times M &\to M \\ (g,x) &\mapsto \Phi_g(x) \end{aligned}$$

$$\begin{aligned} ((a,b),(q^1,q^2,p_1,p_2)) &\mapsto (a+q^1, b+e^a q^2, p_1, e^{-a}p_2) \\ &= \Phi_g(x). \end{aligned} \tag{1}$$

It is easy to check that this is an action. Moreover, since

$$\begin{aligned} & d(a + q^1) \wedge dp_1 + d(b + e^a q^2) \wedge d(e^{-a} p_2) \\ &= dq^1 \wedge dp_1 + dq^2 \wedge dp_2, \end{aligned}$$

this action is actually symplectic (i.e. leaves ω invariant).

Let $\theta = p_1 dq^1 + p_2 dq^2$. Then $\omega = -d\theta$. The action of G also leaves θ invariant. Hence by theorem 4.2.10 (Abraham & Marsden [3]) there is an Ad^* equivariant momentum mapping J, defined by,

$$\begin{aligned} & J : \mathbb{R}^4 \to \Im^* = \mathbb{R}^2 \\ & J(x) \cdot \xi = (i_{\xi_M} \theta)(x). \end{aligned}$$

We now compute J explicitly.

First, the Lie algebra corresponding to G with the Lie bracket $[\cdot,\cdot]$ is

$$\begin{aligned} \Im = \{\xi = (\xi^1, \xi^2) \epsilon \mathbb{R}^2 \mid \\ [\xi, \eta] = (0, \xi^1 \eta^2 - \xi^2 \eta^1)\}. \end{aligned}$$

It follows that for $\xi \epsilon \Im$, the exponential map from $\Im$ to G is given by

$$\exp(t\xi) = \begin{cases} (0, t\xi^2) & \text{if } \xi^1 = 0 \\ (t\xi^1, (e^{t\xi^1} - 1)\frac{\xi^2}{\xi^1}) & \text{if } \xi^1 \neq 0. \end{cases} \tag{2}$$

The adjoint action of G on $\Im$ is given by

$$\begin{aligned} \mathrm{Ad} : G \times \Im &\to \Im \\ (g, \xi) &\mapsto Ad_g(\xi) \\ &= T_e(R_{g-1} L_g)\xi. \end{aligned}$$

In our case, for $g = (a, b)$, $\xi = (\xi^1, \xi^2)$,

$$Ad_g \xi = (\xi^1, e^a \xi^2 - b\xi^1),$$

or

$$Ad_g \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & e^a \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \tag{3}$$

The coadjoint action of G on $\Im^*$ is

$$Ad^* : G \times \Im^* \to \Im^*$$
$$(g, \ell) \mapsto Ad^*_{g-1}(\ell).$$

For $\ell = (\ell_1, \ell_2)$, $g = (a, b)$, we have

$$Ad^*_{g-1}\begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} = \begin{pmatrix}1 & e^{-a}b\\ 0 & e^{-a}\end{pmatrix}\begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix}. \tag{4}$$

The infinitesimal generator of the action corresponding to $\xi = (\xi^1, \xi^2)$ can be obtained as, for $x = (q^1, q^2, p_1, p_2)$,

$$\begin{aligned}\xi_M(x) &= \frac{d}{dt}\Phi(exp\ (t\xi), x)|_{t=0}\\ &= (\xi^1, q^2\xi^1 + \xi^2, 0, -\xi^1 p_2).\end{aligned} \tag{5}$$

We are now ready to compute the momentum mapping J. The computation is as follows.

$$\begin{aligned}J(x)\,(\xi) &= (i_{\xi_M}\theta)(x)\\ &= (p_1 dq^1 + p_2 dq^2)\left(\xi^1\frac{\partial}{\partial q^1} + (q^2\xi^1 + \xi^2)\frac{\partial}{\partial q^2} - \xi^1 p_2\frac{\partial}{\partial p_2}\right)\\ &= p_1\xi^1 + p_2(q^2\xi^1 + \xi^2).\end{aligned}$$

We may then write J as

$$J : \mathbb{R}^4 \to \mathbb{R}^2 = \Im^*$$
$$\begin{pmatrix}q^1\\ q^2\\ p_1\\ p_2\end{pmatrix} \mapsto \begin{pmatrix}p_1 + p_2 q^2\\ p_2\end{pmatrix} \tag{6}$$

We now carry out the Marsden-Weinstein (symplectic) reduction procedure, [2].

First, we choose $\mu = (0,0) \in \Im^*$. By (6), we know that, $J^{-1}(0) = \{(q^1, q^2, 0, 0,)|q^1, q^2 \in \mathbb{R}\}$.

Since the Jacobian matrix of J is

$$DJ = \begin{bmatrix}0 & p_2 & 1 & q^2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

which has rank 2 for all points in $J^{-1}(0)$, it follows that $\mu = (0,0)$ is a regular value of J. Next, we find that the isotropy group at (0,0) is just the whole group G, which can be checked from (4).

$$G_0 = \{g = (a,b) | Ad^*_{g^{-1}}(0) = 0\}$$
$$= G.$$

The action of G_0 on $J^{-1}(0)$ is just the action G on $J^{-1}(0)$.

In order to have a good quotient space, we need to check if the action is free and proper (which implies the action is simple.) Obviously, the map,

$(a,b) \rightarrow (a + q^1, b + e^a q^2, 0, 0)$ is one-to-one. Thus, the action is free. Next, assume that, as $n \rightarrow \infty$,

$(q^1_n, q^2_n, 0, 0 \rightarrow (q^1, q^2, 0, 0)$ and $(a_n + q^1_n, b_n + e^{a_n} q^2_n, 0, 0) \rightarrow (\gamma^1, \gamma^2, 0, 0)$,

Then $$\begin{cases} a_n \rightarrow \gamma^1 - q^1 \\ b_n \rightarrow \gamma^2 - e^{\gamma^1 - q^1} q^2, \end{cases}$$

which shows the action is proper. Now we can apply Thm. 4.3.1. [Abraham and Marsden] to find the symplectic reduced manifold. In fact G_0 acts transitively on $J^{-1}(0)$, and hence P_0 is a 1 point manifold. From the reduction theorem, there exists a unique ω_0 (symplectic form) on P_0 such that

$$\pi_0^* \, \omega_0 = i_0^* \omega$$

where $$\begin{cases} \pi_0 : J^{-1}(0) \rightarrow P_0 & \text{is the canonical projection,} \\ i_0 : J^{-1}(0) \hookrightarrow P & \text{is the inclusion map.} \end{cases}$$

In our case, ω_0 degenerates to 0. Now we consider the dynamics on the manifold. If we define the Hamiltonian function on $\mathbb{R}^4$ by

$$H : \mathbb{R}^4 \rightarrow \mathbb{R}$$
$$(q^1, q^2, p_1, p_2) \mapsto p_2 e^{q^1},$$

It is easy to check that H is G invariant and,

$H = e^{q^1} \frac{\partial}{\partial q^2} - p_2 e^{q^1} \frac{\partial}{\partial p_1}$ is the Hamiltonian vector field on $(\mathbb{R}^4, \omega)$ associated to H.

The flow of X_H is given by,

$$F^t_{X_H}\left((q^1, q^2, p_1, p_2)\right) = (q^1, q^2 + te^{q^1}, p_1 - tp_2 e^{q^1}, p_2) \tag{7}$$

Now, we can apply Theorem 4.3.5. in [2] to find the reduced dynamics as

$$\begin{aligned} H_\mu &= 0, \\ X_{H_\mu} &= 0. \end{aligned}$$

Obviously, the reduced dynamics is trivially Lyapunov stable.

Now we are ready to investigate the notion of relative stability. Before doing that, we note that any point in $J^{-1}(0)$ maps to one point in $P_0 = J^{-1}(0)/G_0$. Hence any point x in $J^{-1}(0)$ is a relative equilibrium, (i.e. the corresponding flow $F^t_{X_H}(x)$ is a stationary motion (in the sense of Libermann-Marle)).

The following computations confirm the argument we have made above.

For $x \in J^{-1}(0)$, we have to find $\xi \epsilon \mathfrak{G}$ such that

$$F^t_{X_H|J^{-1}(0)}(x) = \Phi(exp\ (t\xi), x). \tag{8}$$

In coordinates,

$$F^t_{X_H|J^{-1}(0)}(x) = (q^1, q^2 + te^{q^1}, 0, 0)$$

$$\Phi(exp\ (t\xi), x) = \begin{cases} (q^1 + t\xi^1, e^{t\xi^1}q^2 + (e^{t\xi^1} - 1)\frac{\xi^2}{\xi^1}, 0, 0) & \text{if } \xi^1 \neq 0 \\ (q^1, q^2 + t\xi^2, 0, 0) & \text{if } \xi^1 = 0. \end{cases}$$

If we choose,

$$\xi^1 = 0 \ , \quad \xi^2 = e^{q^1},$$

then (8) is satisfied.

Remark

$M/G \approx \mathbb{R}^2$ is a good quotient.

The G invariant dynamics X_H descends to M/G. We denote the quotient dynamics as $\hat{X}$.

If we denote by $\pi : \ M \to M/G$ the canonical projection, then the following are equivalent characterizations of relative equilibria.

$x \epsilon J^{-1}(\mu)$ is a relative equilibrium

$\leftrightarrow$

$F^t_{X_H}(x) = \Phi(exp\,(t\xi), x)$

for some $\xi \epsilon \Im$

$\leftrightarrow$

$F^t_{X_H}(x)$ is a stationary motion

$\leftrightarrow X_{H_\mu}(\pi_\mu(x)) = 0$

$\leftrightarrow \hat{X}(\pi(x)) = 0$

Definition We say that $F^t_{X_H}(x)$ is <u>relatively stable mod G</u> if $\pi(x)$ is a Lyapunov stable equilibrium point of $\hat{X}$.

Now consider $x = (q^1, q^2, p_1, p_2) = (0,0,0,0) \epsilon \mathbb{R}^4$

$F^t_{X_H}(x) = (0,t,0,0)$ is a stationary motion since $F^t_{X_H}(x) \subset J^{-1}(0)\ \forall\, t \epsilon \mathbb{R}$

<u>Is this motion relatively stable mod G</u>? We can coordinatize M/G via

$$\pi(x) = \begin{pmatrix} p_1 \\ p_2 e^{q^1} \end{pmatrix}.$$

This is because

$$\begin{pmatrix} q^1 \\ q^2 \\ p_1 \\ p_2 \end{pmatrix}$$

is in the G orbit of

$$\begin{pmatrix} 0 \\ 0 \\ p_1 \\ e^{q_1} p_2 \end{pmatrix}.$$

$$\text{Clearly,} \quad \pi\,\{(0,t,0,0,)|t \epsilon \mathbb{R}\} = (0,0)$$
$$= \text{equilibrium of } \hat{X}$$

as it should be.

The question of relative stability mod G of the flow $(0,t,0,0)$ reduces to a question of Lypunov stability of the equilibrium $\pi\,(x) = (0,0) \epsilon\, M/G$.

Choose

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ e^{+q^1} p_2 \end{pmatrix}$$

in a neighborhood of $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \epsilon\, M/G$. Let $x_\lambda = (0, 0, \lambda_1, \lambda_2) \,\epsilon\, \pi^{-1}(\lambda)$.

Then,

$$F^t_{X_H}(x_\lambda) = (0, t, \lambda_1 - t\lambda_2, \lambda_2)$$

$$\pi \circ F^t_{X_H}(x_\lambda) = \begin{pmatrix} \lambda_1 - t\lambda_2 \\ \lambda_2 \end{pmatrix}.$$

Clearly if $\lambda_2 \neq 0$, $\pi \circ F^t_{X_H}(x_\lambda)$ leaves any neighborhood of $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \epsilon\, M/G$ in finite time!

Hence the stationary motion $(0, t, 0, 0)$ is not relatively stable mod G.

References to the Appendix

[1] P. Libermann, C.-M. Marle, "Symplectic Geometry and Analytical Mechanics", D. Reidel Publishing Company 1987.

[2] J.E. Marsden, A. Weinstein, "Reduction of Sympletic Manifolds with Symmetry". Reports on Mathematical Physics. 5. 1974. pp. 121-130.

[3] R. Abraham, J.E. Marsden, "Foundations of Mechanics" 2nd Edition., The Benjamin/Cummings Publishing Company 1978.

Department of Electrical Engineering and Systems Research Center
University of Maryland, College Park
College Park, MD 20742